Mathematical Finance
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Assignment 10

Variance swap hedging by using liquid call options

Fix some horizon $T > 0$. We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{Q})$, where $\mathbb{Q}$ is directly assumed to be a risk-neutral measure, under which the dynamics of the unique risky asset S is given by

$$S_t = S_0 + \int_0^t S_s\sigma_s \mathrm{d}B_s^{\mathbb{Q}},\ t \in [0,T],$$

where $S_0 > 0$, where $B^{\mathbb{Q}}$ is an $(\mathbb{F},\mathbb{Q})$–Brownian motion, and where the process $(\sigma_t)_{t\in[0,T]}$ is an $\mathbb{F}$-adapted process satisfying $\underline{\sigma} \leq \sigma_t \leq \overline{\sigma},\ ,t \in [0,T]$, for some constants $0 < \underline{\sigma} \leq \overline{\sigma} < +\infty$. We assume that the interest rate is constant and equal to 0 for simplicity. The first goal of this exercise is to price and hedge the random leg of a so-called variance swap with maturity T, whose payoff is given by

$$J_T := \frac{1}{T}\int_0^T \sigma_t^2 \mathrm{d}t.$$

1) By using Itô's formula, show that
$$J_T = -\frac{2}{T}\log\left(\frac{S_T}{S_0}\right) + \frac{2}{T}\int_0^T \sigma_t \mathrm{d}B_t^{\mathbb{Q}},$$
and verify that
$$\frac{2}{T}\int_0^T \sigma_t \mathrm{d}B_t^{\mathbb{Q}} = \int_0^T \phi(S_t)\mathrm{d}S_t,$$
for some function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ that you will explicitly determine.

2) Deduce from the previous question a replication strategy for an option with maturity T and with payoff $\frac{2}{T}\int_0^T \sigma_t \mathrm{d}B_t^{\mathbb{Q}}$.

3) We assume in this question that it is possible to sell in the market, at the price p, a European option with maturity T and with payoff $\log(S_T)$ (this is called a *log-contract*). How can you then replicate and hedge J_T? Prove that the price of this hedge is given by
$$\frac{2}{T}\big(\log(S_0) - p\big).$$

4) We assume from now on that we can buy and sell dynamically a European call with maturity $T' > T$ and strike S_0. Explain why its price P_t at any time $t \leq T$ is given by
$$P_t = \mathbb{E}^{\mathbb{Q}}\big[(S_{T'} - S_0)^+ \big|\mathcal{F}_t\big].$$

5) We assume now that the dynamics of σ is of the form
$$\sigma_t = \sigma_0 + \int_0^t \varphi(S_s,\sigma_s)\big(\mathrm{d}B_s^{\mathbb{Q}} + \eta \mathrm{d}W_s^{\mathbb{Q}}\big),$$
where $\sigma_0 > 0$, $\eta > 0$, $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and bounded function, and $W^{\mathbb{Q}}$ is another $(\mathbb{F},\mathbb{Q})$–Brownian motion, independent of $B^{\mathbb{Q}}$. Explain why there exist functions $p : [0,T]\times\mathbb{R}_+\times\mathbb{R}_+$ and $v : [0,T]\times\mathbb{R}_+\times\mathbb{R}_+$ such that
$$P_t = p(t,S_t,\sigma_t),\ \mathbb{E}^{\mathbb{Q}}\big[\log(S_T)\big|\mathcal{F}_t\big] = v(t,S_t,\sigma_t),\ t\in[0,T].$$

6) We assume that the functions p and v of the previous question are smooth. Verify first that if we define the following two-dimensional vectors

$$X_t := \begin{pmatrix} S_t \\ \sigma_t \end{pmatrix}, \text{ and } \mathcal{B}_t^{\mathbb{Q}} := \begin{pmatrix} B_t^{\mathbb{Q}} \\ W_t^{\mathbb{Q}} \end{pmatrix}, \ t \in [0,T],$$

we can then write

$$X_t = \begin{pmatrix} S_0 \\ \sigma_0 \end{pmatrix} + \int_0^t \begin{pmatrix} X_s^1 X_s^2 & 0 \\ \varphi(X_s^1, X_s^2) & \eta\varphi(X_s^1, X_s^2) \end{pmatrix} \mathrm{d}\mathcal{B}_s^{\mathbb{Q}}, \ t \in [0,T],$$

where for $i \in \{1,2\}$, X_t^i denotes the ith coordinate of the vector X_t.

Then, using Itô's formula, prove that for any $t \in [0,T]$

$$\begin{aligned} p(t,S_t,\sigma_t) = p(0,S_0,\sigma_0) &+ \int_0^t \bigg(\frac{\partial p}{\partial t}(s,S_s,\sigma_s) + \frac{1}{2} S_s^2\sigma_s^2 \frac{\partial^2 p}{\partial S^2}(s,S_s,\sigma_s) + \frac{1+\eta^2}{2}\varphi^2(S_s,\sigma_s)\frac{\partial^2 p}{\partial \sigma^2}(s,S_s,\sigma_s)\bigg)\mathrm{d}s \\ &+ \int_0^t S_s\sigma_s\varphi(S_s,\sigma_s)\frac{\partial^2 p}{\partial S \partial \sigma}(s,S_s,\sigma_s)\mathrm{d}s + \int_0^t \bigg(S_s\sigma_s \frac{\partial p}{\partial S}(s,S_s,\sigma_s) + \varphi(S_s,\sigma_s)\frac{\partial p}{\partial \sigma}(s,S_s,\sigma_s)\bigg)\mathrm{d}B_s^{\mathbb{Q}} \\ &+ \int_0^t \eta\varphi(S_s,\sigma_s)\frac{\partial p}{\partial \sigma}(s,S_s,\sigma_s)\mathrm{d}W_s^{\mathbb{Q}}, \end{aligned}$$

and

$$\begin{aligned} v(t,S_t,\sigma_t) = v(0,S_0,\sigma_0) &+ \int_0^t \bigg(\frac{\partial v}{\partial t}(s,S_s,\sigma_s) + \frac{1}{2} S_s^2\sigma_s^2 \frac{\partial^2 v}{\partial S^2}(s,S_s,\sigma_s) + \frac{1+\eta^2}{2}\varphi^2(S_s,\sigma_s)\frac{\partial^2 v}{\partial \sigma^2}(s,S_s,\sigma_s)\bigg)\mathrm{d}s \\ &+ \int_0^t S_s\sigma_s\varphi(S_s,\sigma_s)\frac{\partial^2 v}{\partial S \partial \sigma}(s,S_s,\sigma_s)\mathrm{d}s + \int_0^t \bigg(S_s\sigma_s \frac{\partial v}{\partial S}(s,S_s,\sigma_s) + \varphi(S_s,\sigma_s)\frac{\partial v}{\partial \sigma}(s,S_s,\sigma_s)\bigg)\mathrm{d}B_s^{\mathbb{Q}} \\ &+ \int_0^t \eta\varphi(S_s,\sigma_s)\frac{\partial v}{\partial \sigma}(s,S_s,\sigma_s)\mathrm{d}W_s^{\mathbb{Q}}, \end{aligned}$$

7) Explain why $(p(t,S_t,\sigma_t))_{t\in[0,T]}$ and $(v(t,S_t,\sigma_t))_{t\in[0,T]}$ are $(\mathbb{F},\mathbb{Q})$-martingales, and deduce that

$$\begin{aligned} p(t,S_t,\sigma_t) &= p(0,S_0,\sigma_0) + \int_0^t \bigg(S_s\sigma_s \frac{\partial p}{\partial S}(s,S_s,\sigma_s) + \varphi(S_s,\sigma_s)\frac{\partial p}{\partial \sigma}(s,S_s,\sigma_s)\bigg)\mathrm{d}B_s^{\mathbb{Q}} + \int_0^t \eta\varphi(S_s,\sigma_s)\frac{\partial p}{\partial \sigma}(s,S_s,\sigma_s)\mathrm{d}W_s^{\mathbb{Q}}, \\ v(t,S_t,\sigma_t) &= v(0,S_0,\sigma_0) + \int_0^t \bigg(S_s\sigma_s \frac{\partial v}{\partial S}(s,S_s,\sigma_s) + \varphi(S_s,\sigma_s)\frac{\partial v}{\partial \sigma}(s,S_s,\sigma_s)\bigg)\mathrm{d}B_s^{\mathbb{Q}} + \int_0^t \eta\varphi(S_s,\sigma_s)\frac{\partial v}{\partial \sigma}(s,S_s,\sigma_s)\mathrm{d}W_s^{\mathbb{Q}}, \end{aligned}$$

8) Deduce that in order to replicate the payoff $\log(S_T)$, one should hold at each time $t \in [0,T]$ a quantity Ψ_t of options with payoff $(S_{T'} - S_0)^+$, where

$$\Psi_t := \frac{\frac{\partial v}{\partial \sigma}(t,S_t,\sigma_t)}{\frac{\partial p}{\partial \sigma}(t,S_t,\sigma_t)},$$

as well as Δ_t risky assets S, where

$$\Delta_t := \frac{\partial v}{\partial S}(t,S_t,\sigma_t) + \frac{\varphi(S_t,\sigma_t)}{S_t\sigma_t}\frac{\partial v}{\partial \sigma}(t,S_t,\sigma_t) - \Psi_t\bigg(\frac{\partial p}{\partial S}(t,S_t,\sigma_t) + \frac{\varphi(S_t,\sigma_t)}{S_t\sigma_t}\frac{\partial p}{\partial \sigma}(t,S_t,\sigma_t)\bigg).$$

9) How can we hedge dynamically J_T by using the risky asset and the call (and thus not the *log-contract*)? Prove that the price of this hedge in terms of S_0 and $v(0,S_0,\sigma_0)$ is given by

$$\frac{2}{T}\big(\log(S_0) - v(0,S_0,\sigma_0)\big).$$

What partial differential equation does the function v satisfy?

10) We consider now a so–called *weighted variance swap* with maturity T and continuous weight function $w : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, for which the random leg's payoff is given by

$$J_T^w := \frac{1}{T}\int_0^T \sigma_t^2 w(S_t)\mathrm{d}t.$$

Define for some fixed $x_0 > 0$ the function $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ by $F(x) := \int_{x_0}^x \int_{x_0}^y \frac{2w(z)}{Tz^2}\mathrm{d}z\mathrm{d}y$. Prove that

$$J_T^w = F(S_T) - F(S_0) - \int_0^T F'(S_t)\mathrm{d}S_t.$$

11) How can you hedge J_T^w, without knowing σ, by simply using cash, Call options, Put options and the risky asset S?